Applications of the Fourier transform in the imaging analysis

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Abstract

In this work I intend to emphasize the importance of the Fourier trasform in the imaging analysis, to do so I'll bring some examples of trasformations in the frequency domain.
First of all, I will put the focus on the definition of the Fourier transform in its different notations and its most important properties.
Then I'll bring some examples of the calculation of the operator on simple functions, in both one and two dimensions,through the use of the software Matlab .
Finally I 'll show one of the applications of the transform in the imaging analysis, underlining its advantages over the work in the spatial domain.

Chapter 1
Definition of the Fourier trasform

The Fourier transform is widely used in the analysis of dynamical systems, in solving differential equations and signal theory.
The reason for such a wide use resides in the fact that it is a tool that allows to decompose and subsequently recombine a generic signal into an infinite sum of sinusoids with different frequencies, amplitudes and phases.
The function obtained by the Fourier transform, continuous or discrete, is said amplitude spectrum and represents how wide are the harmonics that compose the original function.

If the signal in question is a periodic signal, its Fourier transform is a discrete set of values, which in this case takes the name of discrete spectrum, and so you need to use the discrete Fourier transform (DTF) to switch from the spatial domain to the frequency one.
Instead, if the function is not periodic the spectrum is continuous and to change the domain you will have to use the integral form of the transform which is defined by:

$$\mathcal{F}\left[f\left(t\right)\right]=F\left(\omega\right)=A\int_{-\infty}^{+\infty}e^{-i\omega t}f\left(t\right)dt \tag{1}$$

with $f\left(t\right)\in L_1\left(-\infty,+\infty\right)$ and $t,\omega\in R$.
Reversing the formula, it can be derived the definition of the inverse Fourier trasform:

$$\mathcal{F}^{-1}\left[F\left(\omega\right)\right]=f\left(t\right)=B\int_{-\infty}^{+\infty}e^{i\omega t}F\left(\omega\right)d\omega \tag{2}$$

These are the general definitions of the forward transform and its inverse, but there are different notations depending on the choice of the constants A, B and the exponential term.
All notations are obviously equivalent but it is worth remembering that different authors may use different choices and so you always need to clear what formula of the Fourier transform you are using.
An exemples is the definition widely used in probability theory:

$$F\left(\omega\right)=\int_{-\infty}^{+\infty}e^{-i\omega t}f\left(t\right)dt \tag{3}$$

$$f\left(t\right)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{i\omega t}F\left(\omega\right)d\omega \tag{4}$$

Another notation is used in quantum mechanics:

$$F\left(\omega\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\omega t}f\left(t\right)dt \tag{5}$$

$$f\left(t\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{i\omega t}F\left(\omega\right)d\omega \tag{6}$$

And at last, the definition most used in imaging analysis:

$$F\left(\nu\right)=\int_{-\infty}^{+\infty}e^{-i2\pi\nu t}f\left(t\right)dt \tag{7}$$

$$f\left(t\right)=\int_{-\infty}^{+\infty}e^{i2\pi\nu t}F\left(\nu\right)d\nu \tag{8}$$

This last notation is obtained by interpreting ω as a pulse, so that $\omega = 2\pi\nu$. The equations of the theory of probability thus become:

$$F(\nu) = \int_{-\infty}^{+\infty} e^{-i2\pi\nu t} f(t)\, dt \tag{9}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i2\pi\nu t} F(\nu)\, 2\pi d\nu = \int_{-\infty}^{+\infty} e^{i2\pi\nu t} F(\nu)\, d\nu \tag{10}$$

Chapter 2
Property of the Fourier trasform

The reason why it is convenient to work in the frequency domain rather than the spatial one, is that the Fourier transform has properties which make certain operations much simpler.
To do an exemple in the imaging analysis, the operation of convolution becomes a simple multiplication when we work in the frequency domain.
The most important properties are the following:

1. **Linearity:**

$$\mathcal{F}\left[af(t) + bg(t)\right] = aF(\omega) + bG(\omega) \tag{11}$$

2. **Complex conjugate:**

$$\mathcal{F}\left[\overline{f(t)}\right] = \overline{F(\omega)} \tag{12}$$

3. **Traslation:**

$$\mathcal{F}\left[f(t - t_0)\right] = F(\omega)\, e^{i\omega t_0} \tag{13}$$

4. **Scaling:**

$$\mathcal{F}\left[f(at)\right] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \tag{14}$$

5. **Derivation:**

$$\mathcal{F}\left[\frac{d^n}{dt^n} f(t)\right] = (-i\omega)^n F(\omega) \tag{15}$$

6. **Convolution:**

$$\mathcal{F}\left[f\left(t\right)*g\left(t\right)\right]=\frac{1}{A}F\left(\omega\right)G\left(\omega\right) \tag{16}$$

Where the convolution is defined by:

$$f\left(t\right)*g\left(t\right)=\int_{-\infty}^{+\infty}f\left(\tau\right)g\left(t-\tau\right)d\tau \tag{17}$$

This operation is widely used in the imaging analysis in its discrete form, for exemples to apply filters: each image is in fact a matrix of numbers representing the brightness of the respective pixels, apply a filter in the spatial domain means actually making the convolution between the image matrix and the one which represents the filter (called kernel).
This is therefore a first great advantage of the Fourier transform, as for a computer, the convolution operation is much more complex than a simple multiplication, especially from a computational point of view: if you must apply filters with large kernel, working in the frequency domain can significantly reduce the execution time of the program.

Chapter 3
Calculation of the Fourier transform in one dimension

Before showing the representation of the Fourier transform of some image, it is helpful to see how to calculate the transform of simple function in one dimension and to do so we will use the notation of probability theory (Equation 3).
The first case is the rectangle function defined by:

$$r(t) = \begin{cases} A & |t| \le L, \\ 0 & |t| > L. \end{cases} \tag{18}$$

If we set A = 1 and L = 2 the function becomes a step of height 1 and width 4:

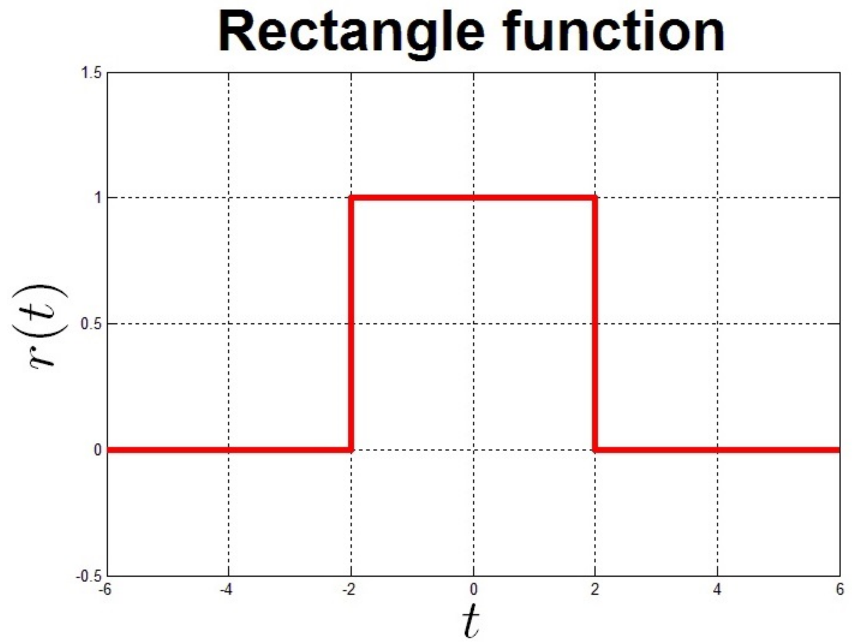

Figure 1: Graphic of the rectangle function, r(t).

The Fourier transform of the rectangle function is:

$$R(\omega) = 2AL \cdot sinc(\omega L) \tag{19}$$

To prove this formula just use the Euler equation and remember that the integral of an odd function over a symmetric range respect to the origin is zero.

$$R(\omega) = \int_{-\infty}^{\infty} r(t) \cdot e^{-i\omega t} dt = \int_{-L}^{L} A \cdot e^{-i\omega t} dt =$$

$$= \int_{-L}^{L} A\left[cos(-\omega t) + isen(-\omega t)\right] dt =$$

$$= A\left\{\int_{-L}^{L} cos(\omega t)dt - i\int_{-L}^{L} sen(\omega t))\right\} =$$

$$= A\int_{-L}^{L} cos(\omega t)dt = A\left[\frac{sen(\omega t)}{wt}\right]_{-L}^{L} =$$

$$= \frac{A}{\omega}\left[sen(\omega L) - sen(-\omega L)\right] = 2AL\frac{sen(\omega L)}{\omega L} =$$

$$= 2AL \cdot sinc(\omega L)$$

The graphic of this result is:

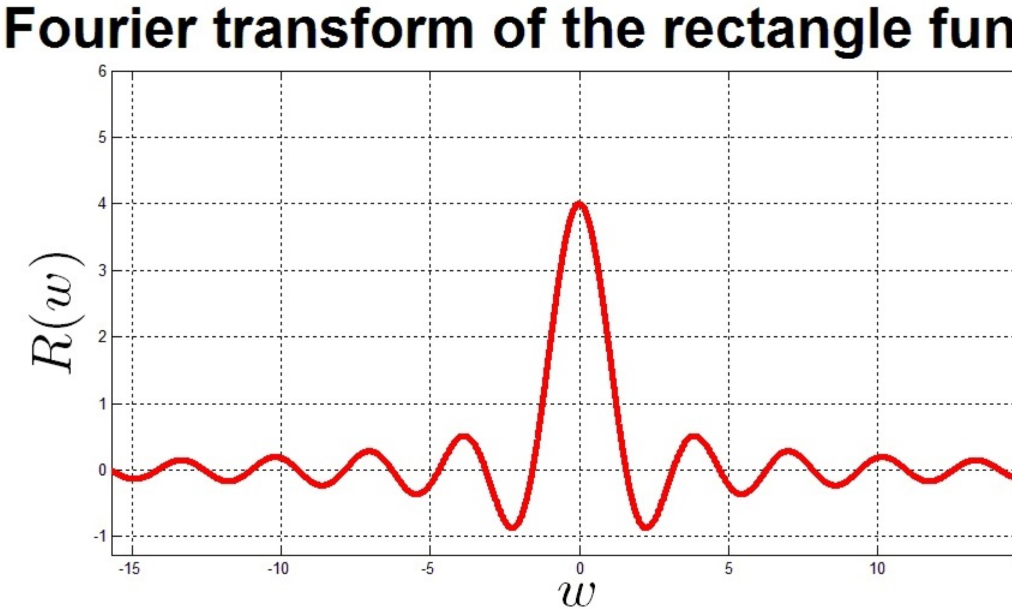

Figure 2: Graphic of the Fourier transform of the rectangle function, R(w).

The zeros of $R(\omega)$ are $\omega = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, ...$ and the central peak height is $R(0) = 2AL$.
If you increase the half width of the rectangle function (L), $R(\omega)$ will become thinner and higher.

The second exemple is the triangle function defined by:

$$f(t) = \begin{cases} A - \frac{A}{L}\,|t| & |t| \le L, \\ 0 & |t| > L. \end{cases} \tag{20}$$

Setting the coefficients A = 1 and L = 2, we obtain:

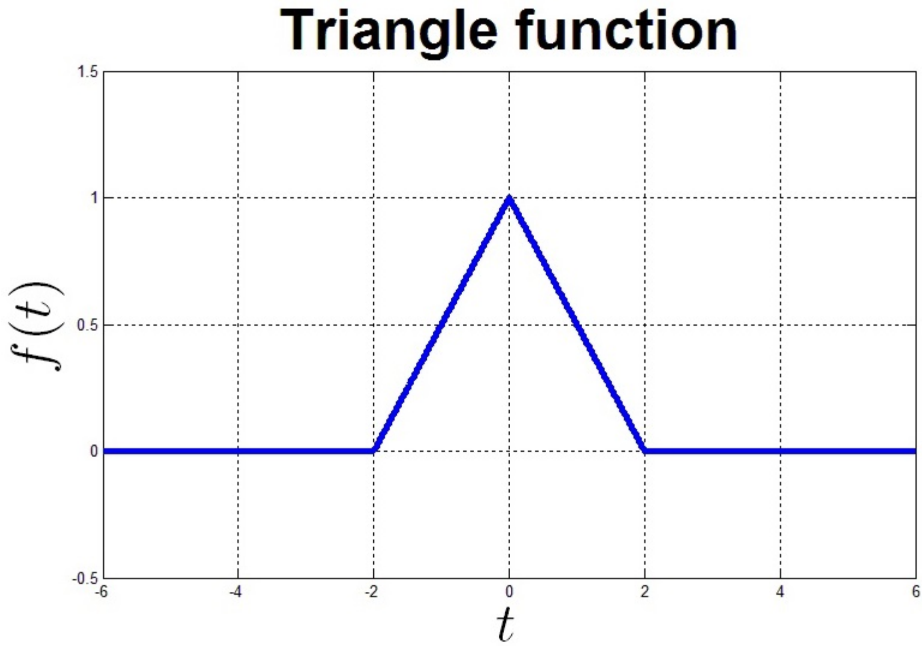

Figure 3: Graphic of the triangle function, f(t).

The Fourier transform of the triangle function is:

$$F(\omega) = AL \cdot sinc^2(\frac{\omega L}{2}) \tag{21}$$

In the demonstration we first divide the integral in the ranges $-L \le t \le 0$ and $0 \le t \le L$ so you do not have the absolute value.
Next, we use integration by parts, the Euler equation and the duplication formula of the cosine function.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt = A\left\{\int_{-L}^{0}\left(1+\frac{t}{L}\right)e^{-i\omega t}dt + \int_{0}^{L}\left(1-\frac{t}{L}\right)e^{-i\omega t}dt\right\} =$$

$$= A\left\{\int_{-L}^{L} e^{-i\omega t}dt + \frac{1}{L}\left\{\int_{-L}^{0} te^{-i\omega t}dt + \int_{0}^{L} te^{-i\omega t}dt\right\}\right\}$$

Using the integration by parts we obtain that:

$$\int te^{-i\omega t}dt = -t\frac{e^{-i\omega t}}{iw} + \frac{e^{-i\omega t}}{\omega^2}$$

Placing this into the equation, it becomes:

$$F(\omega) = A\left\{\left[-\frac{e^{-i\omega t}}{i\omega}\right]_{-L}^{L} + \frac{1}{L}\left\{\left[-t\frac{e^{-i\omega t}}{i\omega} + \frac{e^{-i\omega t}}{\omega^2}\right]_{-L}^{0} + \left[-t\frac{e^{-i\omega t}}{i\omega} + \frac{e^{-i\omega t}}{\omega^2}\right]_{L}^{0}\right\}\right\} =$$

$$= A\left\{\frac{e^{i\omega t}}{i\omega} - \frac{e^{-i\omega t}}{i\omega} + \frac{1}{L}\left\{\frac{1}{\omega^2} - L\frac{e^{i\omega L}}{i\omega} - \frac{e^{i\omega L}}{\omega^2} + \frac{1}{\omega^2} + L\frac{e^{-i\omega L}}{iw} - \frac{e^{-i\omega L}}{\omega^2}\right\}\right\} =$$

$$= \frac{2A}{L\omega^2}\left\{1 - \frac{e^{i\omega L} + e^{-i\omega L}}{2}\right\} = \frac{2A}{L\omega^2}\{1 - cos(\omega L)\}$$

Using the duplication formula of the cosine function:

$$cos(2\alpha) = 1 - 2sen^2(\alpha)$$

We get:

$$F(\omega) = \frac{4A}{L\omega^2}sen^2(\frac{\omega L}{2}) = LA\left\{\frac{sen^2(\frac{\omega L}{2})}{\frac{L^2\omega^2}{4}}\right\} = AL \cdot sinc^2(\frac{\omega L}{2})$$

The graphic of the Fourier transform of the triangle function is:

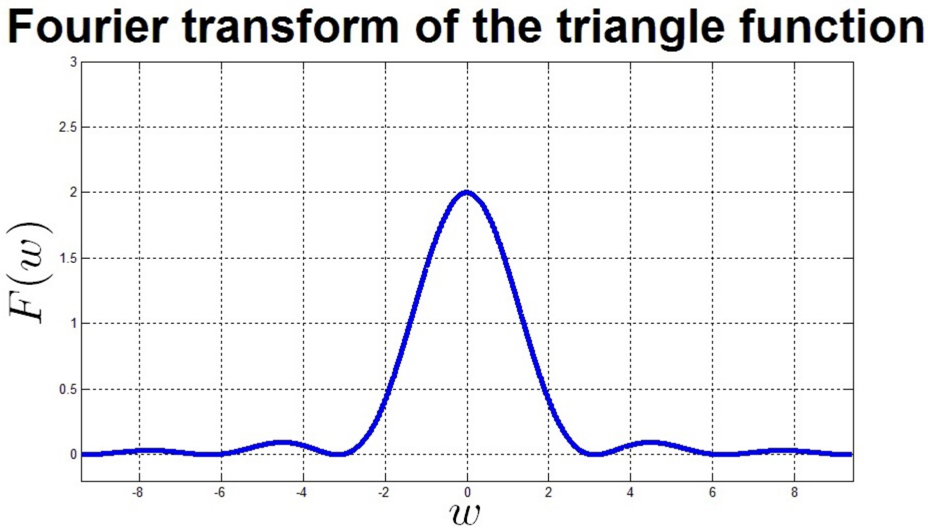

Figure 4: Graphic of the Fourier transform of the triangle function, F(w).

The zeros of $F(\omega)$ are $\omega = \frac{2\pi}{L}, \frac{4\pi}{L}, \frac{6\pi}{L}, ...$ and the central peak height is $R(0) = AL$.
If you increase the parameter L, F(w) will become thinner and higher.
The third and last case that we will see in one dimension is the Gauss function:

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \tag{22}$$

The graphic with $\sigma = 2$ is:

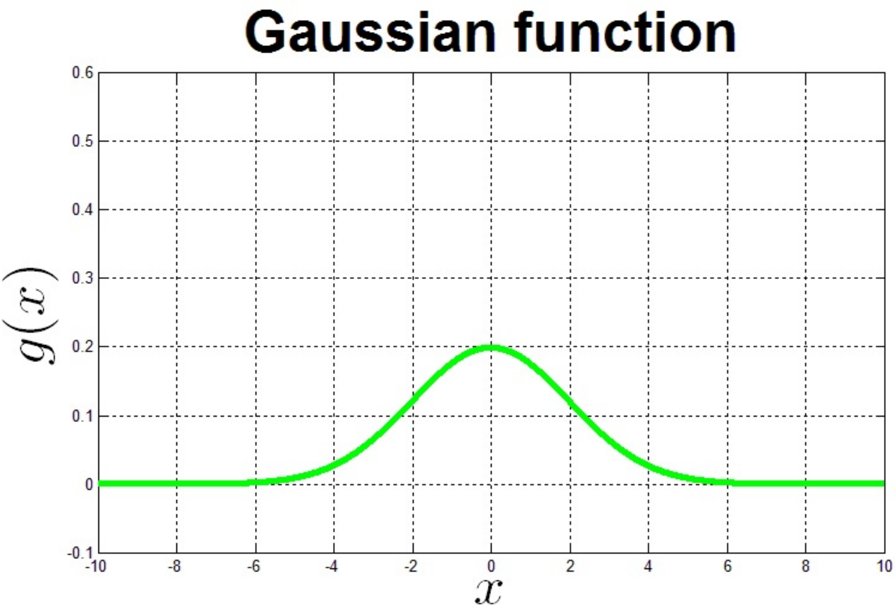

Figure 5: Graphic of the Gauss function, g(x).

The Fourier transform of the Gauss function is:

$$G(k) = e^{-\frac{k^2\sigma^2}{2}} \tag{23}$$

To demonstrate this result we just use the method of completing the square in the exponent:

$$G(k) = \int_{-\infty}^{\infty} g(x)e^{-ikx}dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}-ikx}dx =$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma^2}-ikx-\frac{k^2\sigma^2}{2}+\frac{k^2\sigma^2}{2}\right)}dx =$$

$$= \frac{e^{-\frac{k^2\sigma^2}{2}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}\sigma}+\frac{ik\sigma}{\sqrt{2}}\right)^2}dx =$$

$$= \frac{e^{-\frac{k^2\sigma^2}{2}}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sigma\sqrt{2} \cdot dt = e^{-\frac{k^2\sigma^2}{2}}$$

The graphic of the Fourier transform of the Gauss function is:

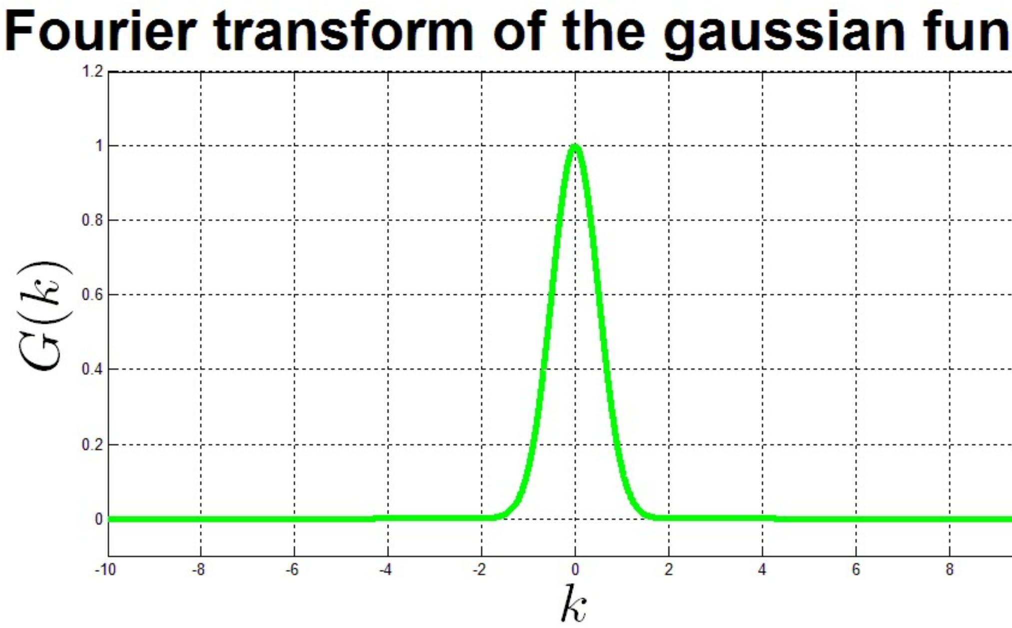

Figure 6: Graphic of the Fourier transform of the Gauss function, G(k).

That is still a Gauss function, but now in the exponent, σ^2 is at the numerator instead of the denominator.
Increasing the parameter, g(x) becomes wider and lower while G(k) becomes thinner and higher.

Chapter 4
Representation of the Fourier transform in two dimensions

The Fourier spectrum of an image is also an image itself, in which pixels, instead of the brightness, there is the amplitude of the harmonic relative to the pixel. The equation used to calculate the transform is:

$$F(u,v) = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} f(x,y) e^{-2\pi i \left(\frac{ux}{M} + \frac{vy}{N}\right)} \tag{24}$$

And its inverse:

$$f(x,y) = \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} F(u,v) e^{2\pi i \left(\frac{ux}{M} + \frac{vy}{N}\right)} \tag{25}$$

The frequencies domain can be represented with different conventions: one of the most used is the one which places the zero at the center of the matrix.

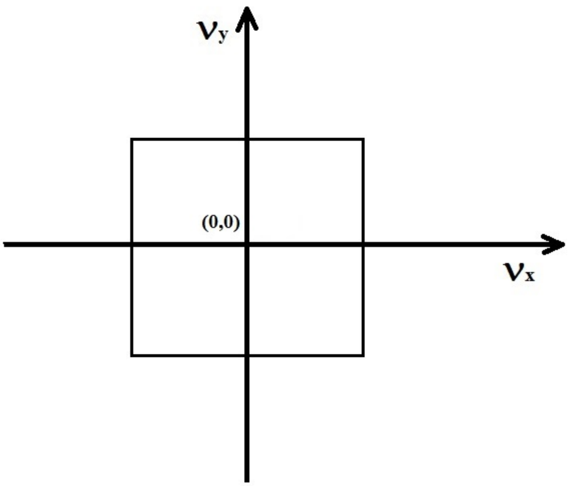

Figure 7: Convention of the axes in the two-dimensional Fourier spectrum.

Because of its complexity, it is very difficult to obtain quantitative information from the display of the Fourier spectrum, however, understanding it may give you a lot of qualitative information that can help working on the image.
Before dealing with complex images, is better to begin with some image sample to learn how to read Fourier spectra.

To display the transformed image it is also necessary to increase contrast as it would otherwise be too dark because of the normalization factor.
Often a linear stretching is not enought because F(0,0), which is the average of the gray levels of the original image f(x,y), is much brighter than the other points.
If for example we create an image with a pattern of sinusoidal brightness along an axis, the Fourier transform will have only three points: the central peak, the point corresponding to the frequency of the function and the point corresponding to the opposite of that frequency.

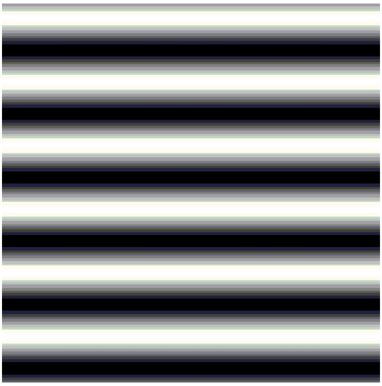
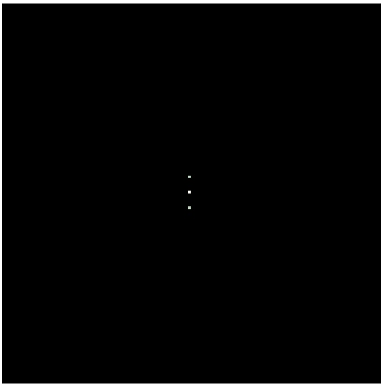

Figure 8: Image of $f(x,y) = sen\left(\frac{2\pi y}{T}\right)$ with $T = 100$ and $y = 1, 2, ..., 600$, and its Fourier spectrum.

If we rorate the image, the transform rotates with the same angle:

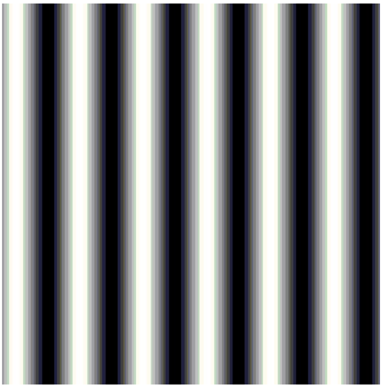
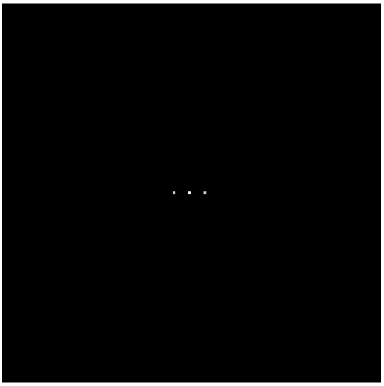

Figure 9: Image of $f(x,y) = sen\left(\frac{2\pi x}{T}\right)$ with $T = 100$ and $x = 1, 2, ..., 600$, and its Fourier spectrum.

Another important example is the transform of a rectangle, in which like the one-dimensional step where you get a sinc, in this case you get two: one along the x-axis and the other along the y-axis.

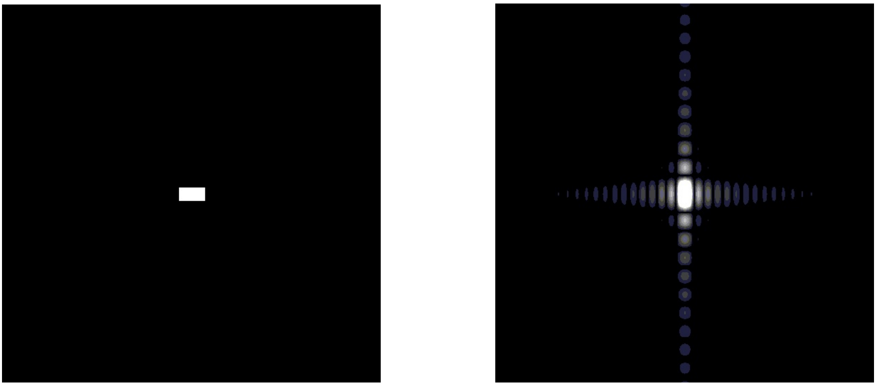

Figure 10: Picture of a rectangle, and its Fourier transform.

To show the last spectrum, it was also necessary to make an absolute value of the resulting image as negative values are not accepted as gray levels.

Chapter 5
Other uses of the Fourier transform in the images analysis

The Fourier spectrum represents therefore the harmonics presence in the original image: according to which areas are lighted you can get an idea of the original image.
The first thing to observe is whitch areas are illuminated: the higher the brightness, the greater the presence of harmonics related to the illuminated pixels.
The second thing is the position of the illuminated zones: if they are close to the center they represent low frequency harmonics, while if they are in boundary regions they stand for high frequency harmonics.
This last aspect is very important because the noise of an image is always a high frequency component, so removing the boundary regions of the spectrum can greatly reduce the noise.
However also the objects of the image have high frequency components: the edges:; therefore the removal of the boundary regions also makes the image blurred (Figure 12).

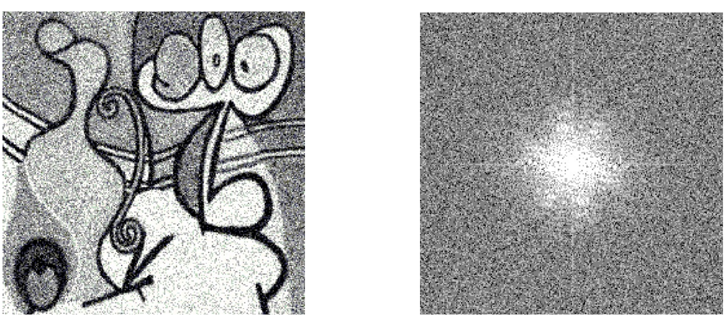

Figure 11: Image with gaussian noise and its Fourier transform.

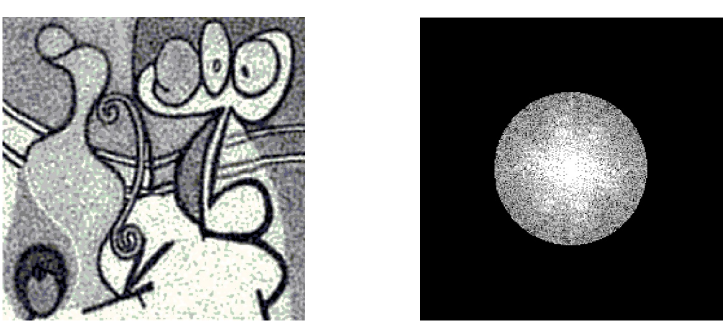

Figure 12: Image 11 with a low pass filter and its Fourier transform.

This type of filter is especially useful when the noise is periodic, since the noise can be distinguished in the transformed image:

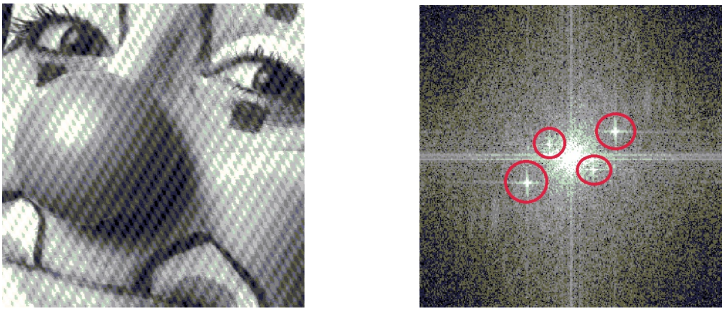

Figure 13: Image with periodic noise and its Fourier transform.

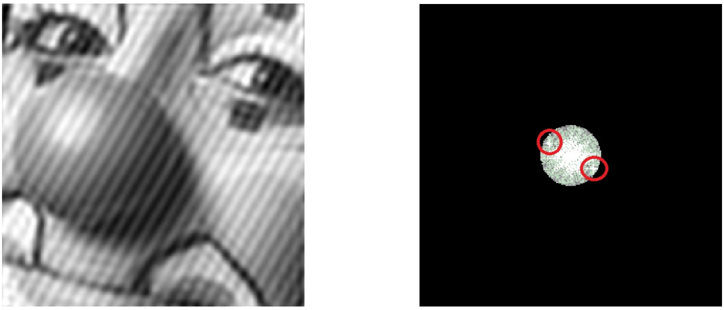

Figure 14: Image 13 with a low pass filter and its Fourier transform.

Figure 15: Image 13 with a low pass filter and its Fourier transform.

The more you cut in the spectrum, the more the image is blurred, and so it may become even worse than the original image.

In Figure 15, you can also see a side effect of the cut in the frequency domain called ringing, which is the generation of false edges close to the real ones and is created by the fact that the cutting was performed with a step function.
For these reasons, the cut of the transform is not used much, while it is preferred to use local cuts or functions more complex for the thresholding, to prevent the creation of artifacts.

Conclusions

In this work we have shown some simple advantages in the use of the Fourier transform in the imaging analysis: the convolution theorem, the reduction of Gaussian noise and the periodic noise suppression. We have also demonstrated that such advantages can however introduce side effects such as blurring or artifacts such as ringing, effects which, however, are easily removed with the use of appropriate filters (for example the edge enhancement filters) or using more complex functions. Nevertheless, this work shows only a framework on the most simple applications of the Fourier transform, the frequency domain is in fact used in many ways in the imaging analysis, such as inverse filtering, analysis of textures and very much more.

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